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any circle V . This circle will belong to the system S, S_1 . From M , the intersection of the radical axis of V and A with the line OO_1 , draw a circle cutting V orthogonally; it will evidently cut S, S_1, A orthogonally, and pass through their points of contact N, N_1 , as in the former case.

Repeating this process with the other three axes of similitude we can determine the other six touching circles.

This solution can be readily extended to include Gen'l Alvord's problem on the tangencies of Spheres.

SUMMATION OF TWO SERIES.

BY PROF. D. TROWBRIDGE, WATERBURGH, NEW YORK.

REQUIRED the sums of the following series:

$$S_x^{(n)} = (a+r)^n + (a+2r)^n + (a+3r)^n + \dots + (a+rx)^n, \quad (1)$$

$$S_x^{(n)} = (a+r)^n + (a+3r)^n + (a+5r)^n + \dots + (a+2rx-r)^n. \quad (2)$$

In these series x is the number of terms, and hence, when $x=0$, the sums will equal nothing. It is plain that the sum of either series will be a function of x ; that is it will depend on x in such a way that when $x=0$, the sum will be 0; and we easily see that the sum will involve the powers of x from x^{n+1} down to x ; we say the $(n+1)$ th power, because the expansion of the last term will give the n th power, and since there are x terms, all similar, there will be the $(n+1)$ th power of x , at least. We may therefore assume

$$(a+r)^n + (a+2r)^n + \dots + (a+rx)^n = A_1 x^{n+1} + A_2 x^n + \dots + A_{n+1} x, \quad (3)$$

$$(a+r)^n + (a+3r)^n + \dots + (a+2rx-r)^n = B_1 x^{n+1} + B_2 x^n + \dots + B_{n+1} x. \quad (4)$$

In these series let $x+1$ be written for x —for they are true for at least all positive integral powers of x —and we shall have

$$(a+r)^n + (a+2r)^n + \dots + (a+rx)^n + (a+rx+r)^n = A_1(x+1)^{n+1} + A_2(x+1)^n + \dots + A_{n+1}(x+1), \quad (5)$$

$$(a+r)^n + (a+3r)^n + \dots + (a+2rx-r)^n + (a+2rx+r)^n = B_1(x+1)^{n+1} + B_2(x+1)^n + \dots + B_{n+1}(x+1). \quad (6)$$

Now substitute (3) from (5) and (4) from (6) and we have

$$(a+rx+r)^n = A_1[(x+1)^{n+1} - x^{n+1}] + A_2[(x+1)^n - x^n] + \dots + A_{n+1}, \quad (7)$$

$$(a+2rx+r)^n = B_1[(x+1)^{n+1} - x^{n+1}] + B_2[(x+1)^n - x^n] + \dots + B_{n+1}. \quad (8)$$

Now make $a+r=b$, and $2r=r'$ in (8) and the two series will have the same form. We can, therefore, easily find the B 's from the A 's.

If we expand the several terms in (7) and equate the coefficients of the like powers of x we can determine the values of $A_1, A_2, \&c.$; and we shall further see that had we assumed a power of x in the sum, greater than $n+1$ its coefficient would have been 0.

We have

$$\begin{aligned}
 (a+rx+r)^n &= (rx+b)^n = r^n x^n + nbr^{n-1}x^{n-1} + \frac{n(n-1)}{1 \cdot 2} b^2 r^{n-2} x^{n-2} + \dots \\
 &= A_1 \left[(n+1)x^n + \frac{(n+1)n}{1 \cdot 2} x^{n-1} + \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3} x^{n-2} + \dots \right] \\
 &\quad + A_2 \left[nx^{n-1} + \frac{n(n-1)}{1 \cdot 2} x^{n-2} + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} x^{n-3} + \dots \right] \\
 &\quad + A_3 \left[(n-1)x^{n-2} + \frac{(n-1)(n-2)}{1 \cdot 2} x^{n-3} + \dots \right] \\
 &\quad + A_4 \left[(n-2)x^{n-3} + \dots \right]. \tag{9}
 \end{aligned}$$

From this equation we have

$$\left. \begin{aligned}
 r^n &= (n+1)A_1, \quad nbr^{n-1} = \frac{(n+1)n}{1 \cdot 2} A_1 + nA_2, \\
 \frac{n(n-1)}{1 \cdot 2} b^2 r^{n-2} &= \frac{(n+1)n(n-1)}{1 \cdot 2 \cdot 3} A_1 + \frac{n(n-1)}{1 \cdot 2} A_2 + (n-1)A_3, \\
 \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} b^3 r^{n-3} &= \frac{(n+1)n(n-1)(n-2)}{1 \cdot 2 \cdot 3 \cdot 4} A_1 \\
 &\quad + \frac{n(n-1)(n-2)}{1 \cdot 2 \cdot 3} A_2 + \frac{(n-1)(n-2)}{1 \cdot 2} A_3 + (n-2)A_4, \&c.
 \end{aligned} \right\} \tag{10}$$

$$\begin{aligned}
 \frac{n(n-1)(n-2) \dots (n-p)}{1 \cdot 2 \cdot 3 \dots (p+1)} b^{p+1} r^{n-p-1} &= \frac{(n+1)n(n-1) \dots (n-p)}{1 \cdot 2 \cdot 3 \dots (p+2)} A_1 \\
 &\quad + \frac{n(n-1) \dots (n-p)}{1 \cdot 2 \cdot 3 \dots (p+1)} A_2 + \dots + (n-p)A_{p+2}. \tag{11}
 \end{aligned}$$

From these equations we find

$$\left. \begin{aligned}
 A_1 &= \frac{r^n}{n+1}, \quad A_2 = \frac{r^{n-1}}{2} \left[2b - r \right], \quad A_3 = \frac{nr^{n-2}}{12} \left[6b^2 - 6br + r^2 \right], \\
 A_4 &= \frac{n(n-1)b r^{n-3}}{12} \left[2b^2 - 3br + r^2 \right], \&c.
 \end{aligned} \right\} \tag{12}$$

If we now substitute $a+r$ for b we shall have for the sum of series (1)

$$\begin{aligned}
 S_x^{(n)} &= \frac{r^n}{n+1} x^{n+1} + \frac{r^{n-1}}{2} (2a+r)x^n + \frac{nr^{n-2}}{12} (6a^2 + 6ar + r^2)x^{n-1} \\
 &\quad + \frac{n(n-1)a^2 r^{n-3}}{6} (a+2r)x^{n-2} + \dots \tag{13}
 \end{aligned}$$

If we now substitute $a+r$ for b and $2r$ for r in (12) we shall find for the sum of series (2)

$$S_x^{(n)} = \frac{(2r)^n}{n+1} x^{n+1} + (2r)^{n-1} ax^n + \frac{n(2r)^{n-2}}{6} (3a^2 - r^2) x^{n-1} \\ + \frac{n(n-1)(2r)^{n-3}}{6} a (a^2 - r^2) x^{n-2} + \dots \quad (14)$$

If we make $a = 0$ and $r = 1$, we shall have

$$1^n + 2^n + 3^n + \dots + x^n = \frac{1}{n+1} x^{n+1} + \frac{1}{2} x^n + \frac{n}{12} x^{n-1} + \dots \quad (15)$$

$$1^n + 3^n + 5^n + \dots + (2x-1)^n = \frac{2^n}{n+1} x^{n+1} - \frac{n \cdot 2^{n-3}}{3} x^{n-1} + \dots \quad (16)$$

If we make $a = 0$ and $r = 2$ in series (1) we shall have

$$2^n + 4^n + 6^n + \dots + (2x)^n = \frac{2^n}{n+1} x^{n+1} + 2^{n-1} x^n + \frac{n \cdot 2^{n-2}}{3} x^{n-1} + \dots$$

REVISED SOLUTION OF PROBLEM 218.

EDITOR ANALYST:

MR. MEECH, the ingeneous proposer of problem 218, having furnished me with the data from which that question was constructed, and requested me to make a general solution, under fuller conditions, for publication in the *ANALYST*, I hereby cheerfully comply with his request.

GEORGE EASTWOOD.

The problem, under its new aspect, may be stated as follows:

Required the separate rates of dividend of two insolvent estates connected as follows:

JOHN DOE'S ESTATE. Direct liabilities = λ ; his endorsements for Richard Roe = λ_1 less a first dividend on the same to be paid out of Roe's estate. His net assets = α to be increased by dividend on account, = β' , due from Roe's Estate.

RICHARD ROE'S ESTATE. Direct liabilities = λ' ; his endorsements for John Doe = λ_2 , less a first dividend on same to be paid out of Doe's estate. His net assets = α' to be increased by dividend on account, = β , due from Doe's estate.

We have Doe's direct liabilities = λ ; his endorsements = λ_1 ;

Roe's direct liabilities = λ' ; his endorsements = λ_2 ;

Doe's net assets = α ;

Roe's " " = α' ;

Doe's account due from Roe's estate = β' ;

Roe's " " " Doe's " = β .